Finite Difference Approximations and Numerical Greens Functions

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Chapter 1

Finite Differences and Differential Equations

§1.1. Finite Difference Approximation.

The finite difference approximation of a differential equation is based on the idea that the derivative of a continuous-valued function, being itself a continuous-valued function, can be approximated near some specific point by the ratio of differences of the function values near that point and the independent variable values near that point

$$\frac{dy}{dx} \approx \frac{y_1 - y_0}{x_1 - x_0} \tag{1.1.1}$$

By replacing the derivatives in the expression for the differential equation with a ratio of differences we obtain an algebra problem to be solved for some discrete (or sampled) function value, y_j, in terms of neighboring values y_{j+1}, y_{j+2}, ..., y_{j-1}, y_{j-2}, ..., such that

$$y_j = a_{j-1}y_{j-1} + a_{j-2}y_{j-2} + \ldots + b_{j+1}y_{j+1} + b_{j+2}y_{j+2} + \ldots \tag{1.1.2}$$

where the expression in (1.1.2) is solved simultaneously at each point in the numerical model.

This can be seen in detail in the well-known works on numerical analysis. The continued reiteration of the resulting expressions may not converge (to a unique solution) or may not have enough accuracy to be useful. Hence, the major work in finite differencing centers on which approaches converge and predicting the accuracy of the outcome.

The subject of numerical integration appears to be an art rather than a rigorous branch of mathematics. The literature traditionally contains various separate but related topics: interpolation and extrapolation, zeros of polynomials, matrices, numerical quadrature, etc. The numerical solution of differential equations has evolved into two separate areas of study. Ordinary differential equations are widely solved numerically in control systems engineering. The numerical solution of partial differential equations has remained the arcane, albeit moribund, foundation of the generally accepted finite element algorithm used in aircraft engineering. A unifying concept for numerical integration that can include both ordinary differential equations (ODE) and partial differential equations (PDE), if it exists, has not been generally accepted.

§1.2. The Green's Function Comparison.

The subject of Green's functions is almost never discussed as an approach to the numerical solution of differential equations. More of an academic subject, Green's functions are typically studied in the mathematical physics works on electromagnetism or heat diffusion. An interesting aspect of this subject that goes unmentioned is that Green's functions have been found for both oridinary differential equations and partial differential equations. Perhaps due to the segregation between the academic cultures of physics, applied mathematics, and engineering the subject of Green's functions has not attracted the amount of attention given to other techniques.

In the area of vector calculus the impulse response solution is synthesized from Green's Theorem as a Green's function solution. The solution of a differential equation in $U(x)$ is written in terms of the Green's function and its gradients

$$G(x, x'), \frac{dG(x, x')}{dx'} \tag{1.2.3}$$

as a surface integral over the boundary weighted by dG/dx' and a volume integral of the inhomogeneous term weighted by $G(x, x')$.

$$U(x) = -\oint_S \frac{dG(x, x')}{dx'} U(x')dx' + \int_V G(x, x')f(x')dx' \tag{1.2.4}$$

One immediate observation that can be made about the above expression is that the integrals can be interpreted as the continuous, infinitesimal equivalents to sums in the finite domain.

Proposition 1.2.1.

The numerical solution to the simplest homogeneous differential equation, Laplace's equation, in a small region is the weighted sum of the neighboring function samples.

That is to say, the algebraic expression we obtain from "replacing" the continuous-valued derivatives with finite differences is actually an interpolation of the neighbor values and, possibly, perturbed by a source term. The coefficients of the neighbor values are interpolation functions (i.e., influence functions) in the same sense as the Lagrange interpolation formula.

$$U(x') = \sum_{j=1}^{n} L_j(x)U_j + P(x')f(x') \tag{1.2.5}$$

The implication here is that the two terms in equation (1.2.5) correspond to the two terms in the Green's function magic rule, (1.2.4). And furthermore, the numerical equivalent of the Green's function gradient, $dG(x, x')/dx'$, is the interpolation function at the neighbor point j. The numerical equivalent of the Green's function itself is the factor $P(x')$, which is a synthesis of the interpolation functions. More precisely, the factor P(x') would be the numerical Green's function, but here the source and observation locations coincide. In order for the factor P to be named a Green's function it must be capable of propagating the source effect at x' to the function value at x. P must be made a function of both x' and x:

$$P(x') \longrightarrow P(x, x')$$

For the purposes of numerical integration that consideration may or may not be needed. The factor $P(x')$ is used to invert the effect of the source term at x' to find the function value, U, at x'.

§1.3. The Role of Interpolation.

The importance of interpolation in numerical integration is well known. However, it may be useful to view interpolation as the central concept of numerical integration. Consider the following observation taken from the calculus of finite differences (Boole):

PROPOSITION 1.3.2.

The Nth-order finite difference, equivalent to differencing the differential equation $d^n y/dx^n$, is a generating function for producing interpolation functions.

In other words, the algebraic expression obtained from replacing the Nth-order derivative with differences in equation (1.1.1) is an interpolation. The coefficients of the neighboring function values are interpolation functions. It may be worthwhile, in finding support for this idea, to review the D operator in George Boole's 1860 book, *A Treatise on the Calculus of Finite Differences*.

If we entertain the ideas in Proposition 1.2.1 and Proposition 1.3.2 we must include the general problem of an arbitrary differential equation, which is the sum of derivative terms in decreasing order. Consider first the highest-order derivative, and let its solution be a weighted sum of neighbor values and interpolation functions. What will be the effect of adding the lower-order derivative terms? The following proposition is an attempt to provide one answer to that question:

PROPOSITION 1.3.3.

The sum of decreasing-order derivatives in a finite difference equation contribute adjustments to the interpolating functions obtained from the highest-order derivative.

In other words, given an arbitrary differential equation approximated by a finite difference solution, the solution is still a weighted sum of the boundary neighbor points. However, the interpolation functions that would have resulted from only the highest derivative must be modified to satisfy the additional requirement of the lower derivatives.

§1.4. Integration by Terms.

We discuss the finite difference approach as an approximation, and we say that the derivatives are "replaced" by their finite difference equivalents. But this is merely common terminology and is used for convenience. In the actual calculation of a solution it would be dangerous to overlook the fact that a derivative is not "replaced" by a finite difference but rather, it is successively integrated. The first-order difference is obtained from the first-order derivative by The Fundamental Theorem of Calculus:

$$y_1 - y_0 = \int_0^1 \frac{dy}{dx'} dx' \tag{1.4.6}$$

We must assume that $y(x)$ is an analytic function over the region, and so it has a convergent Taylor series at x, and so over a smaller and smaller region its geometric curve approaches a straight line.

The reason for emphasizing this obvious fact is that the reduction of a differential equation to a finite difference should proceed according to a linearized integration over a differential region and not by trivial

substitution by differences. For example, one of the terms of the differential equation could be a parameter that is a function of x, say $b(x)$. Its contribution to the difference equation would be a quadrature:

$$\int_0^1 b(x')dx' \approx \frac{1}{2}(b_1 + b_0) \tag{1.4.7}$$

The quadrature might be the average shown above, or it might be more useful to obtain a higher-order quadrature. The proper decision may be guided in light of Proposition 1.3.3. If we design a finite difference solution to be a weighted sum then the result of equation (1.2.5) could be adjusted to that end. Essentially, a weighted sum is usually defined such that the sum of the weights is unity. During the reduction to finite differences the resulting interpolation functions should be checked that they sum to unity. Additionally, the resulting interpolation functions should be considered influence functions. That is, the weight function of each neighbor value should be unity at that point and be zero at all other neighbor points.

Chapter 2

Interpolating Polynomials and Influence Functions

§2.1. Polynomials in 1 Dimension.

For a more detailed, introductory account of polynomial interpolation see *Elementary Theory and Application of Numerical Analysis* by David G. Moursund and Charles S. Duris, and *Numerical Methods for Scientists and Engineers*, second edition, by R. W. Hamming.

Polynomial interpolation is the operation of finding an nth-order polynomial, $P_n(x)$, that passes through $N+1$ known function points, $f(x_1)$, $f(x_2)$, ..., $f(x_n)$, $f(x_{n+1})$. The interpolating polynomial is known to be solved by two different but equivalent approaches: the Lagrange Interpolation formula, and the Vandermonde matrix. The first of these is the Lagrange Interpolation formula, written

$$P_n(x) = \sum_{j=0}^{n} L_j(x) f(x_j) \tag{2.1.1}$$

where the coefficients, $L_j(x)$, are the *Lagrange interpolation functions*. The interpolation functions are themselves polynomials, each multiplying a corresponding data point, $f(x_j)$, of the data set to be "fitted." The property of each Lagrange function, $L_j(x)$, is that it is unity at the location x_j corresponding to the data point $f(x_j)$ which it multiplies, and is zero at all other data points, $f(x_k), k \neq j$. In other words, this polynomial should be trivial to derive, because the zeros of the polynomial are known. As an example, consider an interpolating polynomial between three points, $f(x_1)$, $f(x_2)$, and $f(x_3)$. To find the Lagrange function $L_3(x)$ that multiplies the point at x_3 we need only construct a polynomial whose zeros are at x_1 and x_2.

$$\overline{L}_3(x) = (x - x_1)(x - x_2)$$

And to make it be unity at x_3 simply calculate the value at $x = x_3$ and divide by the result,

$$L_3(x) = \frac{\overline{L}_3(x)}{\overline{L}_3(x_3)} = \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}$$

The second approach is to obtain the interpolating polynomial from the solution of the Vandermonde matrix. For a set of $N + 1$ data points construct an Nth-degree polynomial

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \tag{2.1.2}$$

whose coefficients are to be solved by matching the polynomial at the $N + 1$ data points. The coefficients a_j are the solution to a set of simultaneous equations, equating the polynomial at each of the $n + 1$ known function values.

$$
\begin{aligned}
P_n(x_1) &= a_0 + a_1 x_1 + a_2 x_1^2 + \ldots + a_n x_1^n \\
P_n(x_2) &= a_0 + a_1 x_2 + a_2 x_2^2 + \ldots + a_n x_2^n \\
&\vdots \\
P_n(x_n) &= a_0 + a_1 x_n + a_2 x_n^2 + \ldots + a_n x_n^n
\end{aligned}
$$

which is equivalent to the matrix equation

$$
\begin{pmatrix}
1 & x_1 & x_1^2 & x_1^3 & \ldots & x_1^n \\
1 & x_2 & x_2^2 & x_2^3 & \ldots & x_2^n \\
 & & \vdots & & & \\
1 & x_n & x_n^2 & x_n^3 & \ldots & x_n^n
\end{pmatrix}
\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix}
=
\begin{pmatrix} P_n(x_1) \\ P_n(x_2) \\ \vdots \\ P_n(x_n) \end{pmatrix}
\tag{2.1.3}
$$

The matrix on the left-hand side of (2.1.3) is the Vandermonde matrix.

Another way to view these two different approaches is that the interpolating polynomial can be constructed from a set of *influence functions* (i.e., the Lagrange functions) that are like propagators, each contributing the influence of the data point it multiplies. The polynomial found by solving the Vandermonde matrix equation (2.1.3) can be likewise constructed by using the Vandermonde matrix equation at each of the data points individually. Say, for the influence function at x_1 the equation would be

$$
\begin{pmatrix}
1 & x_1 & x_1^2 & x_1^3 & \ldots & x_1^n \\
1 & x_2 & x_2^2 & x_2^3 & \ldots & x_2^n \\
 & & \vdots & & & \\
1 & x_n & x_n^2 & x_n^3 & \ldots & x_n^n
\end{pmatrix}
\begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{pmatrix}
=
\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}
\tag{2.1.4}
$$

to obtain the influence function, $L_1(x)$,

$$
L_1(x) = b_0 + b_1 x + b_2 x^2 + \ldots + b_n x^n \tag{2.1.5}
$$

§2.2. Polynomials in 2 Dimensions.

Suppose we want an interpolating polynomial between three points in the plane, x_1, y_1, x_2, y_2, and x_3, y_3. We can follow the same recipe of matching polynomial coefficients by setting values at the three points. The polynomial

$$
P(x, y) = ax + by + c \tag{2.2.6}
$$

is found using the set of simultaneous equations

$$
\begin{pmatrix}
x_1 & y_1 & 1 \\
x_2 & y_2 & 1 \\
x_3 & y_3 & 1
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \end{pmatrix}
=
\begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}
\tag{2.2.7}
$$

The solution is

$$a = \frac{P_1(y_3 - y_2) + P_2(y_1 - y_3) + P_3(y_2 - y_1)}{x_1(y_3 - y_2) + x_2(y_1 - y_3) + x_3(y_2 - y_1)}$$

$$b = -\frac{P_1(x_3 - x_2) + P_2(x_1 - x_3) + P_3(x_2 - x_1)}{x_1(y_3 - y_2) + x_2(y_1 - y_3) + x_3(y_2 - y_1)}$$

$$c = \frac{P_1(x_3y_2 - x_2y_3) + P_2(x_1y_3 - x_3y_1) + P_3(x_2y_1 - x_1y_2)}{x_1(y_3 - y_2) + x_2(y_1 - y_3) + x_3(y_2 - y_1)} \tag{2.2.8}$$

This is a linear, two-dimensional interpolation in x and y. The first observation we make is how easy this formula was to generalized from the one-dimensional case of polynomial interpolation. In the Vandermonde matrix, the y coordinates were simply added to each simultaneous equation. It would seem that the above result provides a more direct alternative to the more common bilinear interpolation method. The points did not need to be arranged with rectangular grid spacing, which leads to the following observation:

PROPOSITION 2.2.1.

Interpolation between points of a scalar field depends on distance, not direction.

Clearly, the above two-dimensional interpolation can also be decomposed into a set of influence functions that weight the data values in the calculation. All that is needed is to recalculate the Vandermonde matrix to be unity at the coordinate x_j, y_j and zero at all others. Although it seems obvious from the examples given, it would be worthwhile to remember the following common property of these interpolations:

PROPOSITION 2.2.2.

Interpolation can be written as a weighted sum.

§2.3. Visual Interpretation in 2 D.

The calculation in equation (2.2.8) becomes impractical, even on a computer, for N greater than 3. The need for more efficient methods for calculating interpolation coefficients leads to a re-examination of polynomial interpolation from a geometric perspective. Your mathematical intuition will tell you that the above formula appears to contain cross products. It fact, the denominator is the cross product of the vectors $\vec{R}_{21}$ connecting points 2 to 1 and vector $\vec{R}_{32}$ connecting point 3 to point 2. In other words, the denominator is equal to twice the area of the triangle. But the interpolation is a weighted sum, and the normalization of the weighted sum is usually the sum of the weights. So, immediately we should expect that the weights are themselves areas.

To see that the influence functions are areas consider the simple example when the triangle encloses the origin. Now each corner point of the triangle can be visualized as a vector from the origin to the point. For example, point x_1, y_1 is a vector $\vec{R}_1$, and so on. Now suppose that we choose to calculate the value at the origin itself. Then $x = 0$, $y = 0$, and the only term remaining to calculate is the coefficient c. Rewriting c as

$$c = P_1\frac{(x_3y_2 - x_2y_3)}{2A} + P_2\frac{(x_1y_3 - x_3y_1)}{2A} + P_3\frac{(x_2y_1 - x_1y_2)}{2A}$$

Noting that the numerators of the terms are $\vec{R}_2 \times \vec{R}_3$, $\vec{R}_3 \times \vec{R}_1$, and $\vec{R}_1 \times \vec{R}_2$, it becomes evident that *the coefficient multiplying each point is twice the area of the triangular sector opposing it.* Then the coefficient c can be rewritten as

$$c = P_1\frac{A_{23}}{A} + P_2\frac{A_{13}}{A} + P_3\frac{A_{12}}{A} \tag{2.3.9}$$

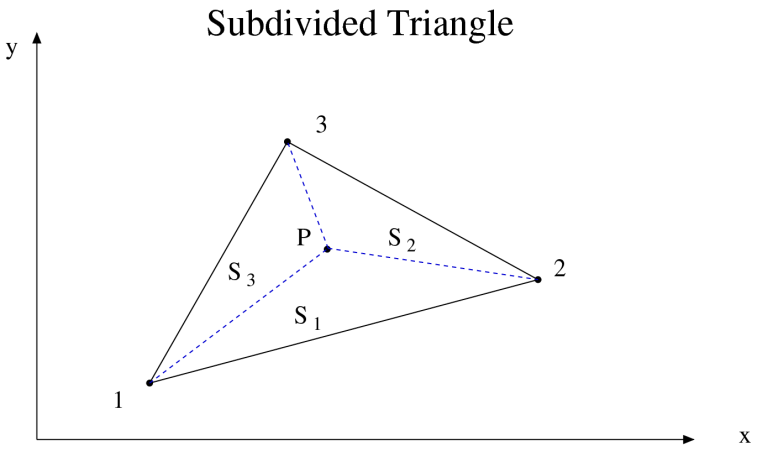

Figure 2.1.

Although we set the coordinates x and y to zero, the result still holds when they are nonzero. The coefficients a and b are offsets to the calculation when x and y are not at the origin.

There is a parallel here to the Lagrange interpolation formulas in one dimension. The analogous linear case in one dimension is just a line segment divided into two segments. The coefficients, i.e., influence functions for the line segments are *the lengths of the opposing line segments.*

§2.4. 2 D Interpolation by Opposing Areas.

The area opposite to the anchoring boundary point is the minimum dimensional quantity which can make the influence function vary between 1 at the anchor point and 0 at all other points, *and goes to zero at the boundary lines.* For polygons with the number of vertices, N, greater than three an opposing area can no longer be uniquely identified.

Consider a four-sided polygon with an observation point x, y in its interior. The line segments between the observation point and the vertices subdivide the polygon into four triangles, S_1, S_2, S_3, and S_4.

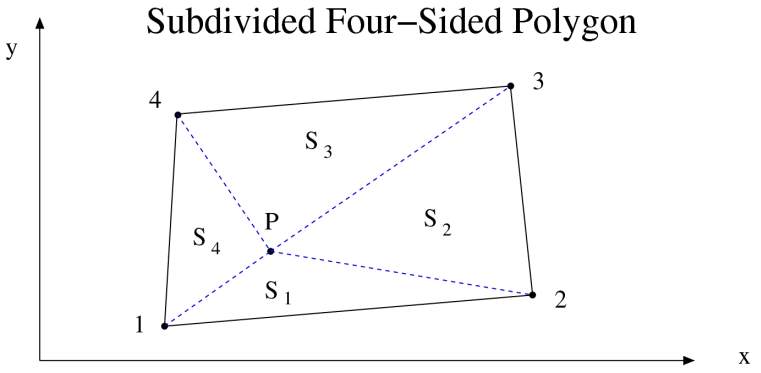

Figure 2.1.

For the influence function corresponding to vertex 1 a function is needed which becomes unity at point 1 and goes to zero at the other three points. For the simpler case of the triangle we could see that each point could be matched to a corresponding sector opposite to the interior point, P. But with a four-sides figure there is

no one sector area that will go to zero when point P moves to all three other points. On the other hand, the *product* of the areas of sectors S_2 and S_3 does go to zero as desired, and becomes unity (when normalized) at point 1. Then similar products can be made to produce influence functions for the other points.

For a five-sided polygon the product of the areas of two sectors is not sufficient. However, the product of three areas will work.

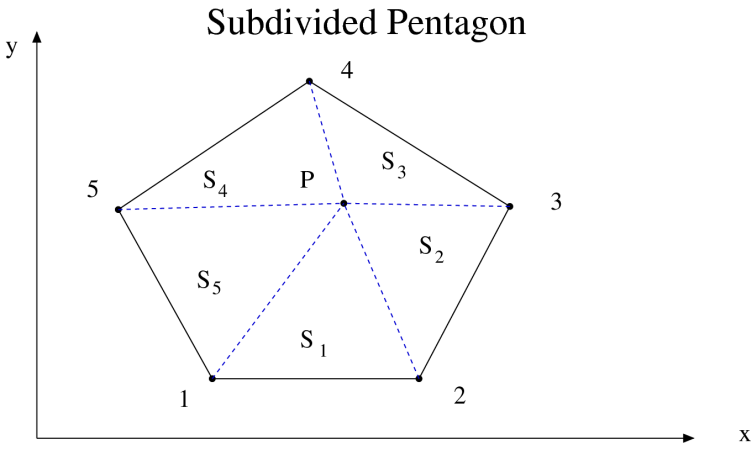

Figure 2.2.

In Figure 2.4.2 it will be observed that as point P is moved to any of vertices 2, 3, 4, or 5 that the product of area sectors S_2, S_3, and S_4 goes to zero. When P moves to vertex 1 the (normalized) product goes to unity. Then for each other vertex in the figure, to construct an influence function which goes to unity for that point and zero for all others it is sufficient to take the product of the sector areas which are not adjacent to it. So, the calculation of the unnormalized influence coefficients becomes

$$\begin{aligned} N_1 &= A_2 A_3 A_4 \\ N_2 &= A_3 A_4 A_5 \\ N_3 &= A_4 A_5 A_1 \\ N_4 &= A_5 A_1 A_2 \\ N_5 &= A_1 A_2 A_3 \end{aligned} \tag{2.4.10}$$

The influence functions are normalized by dividing by the sum of all the weights such that they sum to unity. The general interpolation of the value of U at some point x, y in the pentagon is then calculated by calculating each of the sector areas, A_1, A_2, A_3, A_4, A_5, which are made by drawing lines between the point x, y and each of the vertices, and weighting the values U_1, U_2, U_3, U_4, U_5, with a corresponds product of opposing areas divided by the normalization factor.

$$U(x,y) = \frac{U_1 A_2 A_3 A_4 + U_2 A_3 A_4 A_5 + U_3 A_4 A_5 A_1 + U_4 A_5 A_1 A_2 + U_5 A_1 A_2 A_3}{A_2 A_3 A_4 + A_3 A_4 A_5 + A_4 A_5 A_1 + A_5 A_1 A_2 + A_1 A_2 A_3} \tag{2.4.11}$$

The surface plot of the pentagon with influence function for vertex 1 set to unity is shown in the figure below. The advantage of interpolation by opposing areas is that the influence function for a given point goes to zero, not only at the zero points, but along the side curves between the zero points.

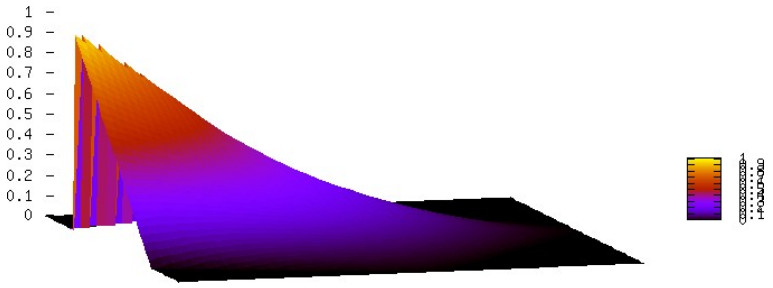

Figure 2.3.

By comparison, observe the corresponding influence function in the following figure which was derived from a Vandermonde matrix solution. Note that the curves between the zero points oscillate.

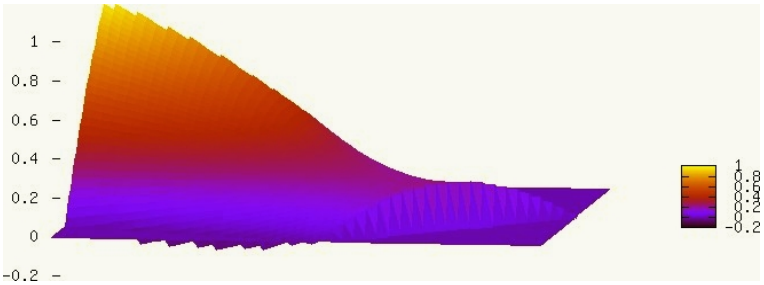

Figure 2.4.

Chapter 3

Introduction to Green's Functions

§3.1. Definition of Green's Functions.

Green's functions are a form of impulse response function in the study of vector calculus. Applications of Green's functions are found in mathematical physics topics such as electromagnetism, heat transfer, fluid dynamics, and quantum mechanics. This section is intended only as the barest introduction. For rigorous derivations of Green's functions see Arfken, "Mathematical Methods for Physicists, and Jackson, "Classical Electrodynamics."

The solution of a differential equation is composed of the homogeneous solution and an inhomogeneous term. The Greens function is the inhomogeneous part of the solution to differential equation as the sum of impulse forces (i.e. sources). If you write an arbitrary differential equation in $y(x)$ as an operator, $\mathcal{L}$, acting on $y(x)$,

$$\mathcal{L} \cdot y(x) = -f(x) \tag{3.1.1}$$

Then the inhomogeneous part of the solution (i.e., the particular solution) is written

$$y_p(x) = \int_{x'} G(x, x') f(x') dx' \tag{3.1.2}$$

This is the superposition of the forcing (source) term in the differential equation acting as an impulse at each location, x, multiplied by the *propagator* function $G(x, x')$. The function $G(x, x')$ is the effect at x due to the force, $f(x)$, acting at x'. The defining property of $G(x, x')$ is that when the differential equation operator, $\mathcal{L}$, acts on G the result is zero everywhere but the location of the impulse force, i.e.,

$$\mathcal{L} \cdot G(x, x') = -\delta(x - x') \tag{3.1.3}$$

The delta function on the right-hand side is the *Dirac delta function*. It is an impulse function defined to be zero everywhere that its argument is nonzero. And where its argument is zero its value is "infinite," or said more accurately, undefined. The Dirac delta function is obtained by differentiating a step function.

The above three equations provide the simplest definition of the Greens function for one-dimensional differential equations. In two and three dimensions the above equations are modified by the use of vector calculus operators, but the idea is the same. A common application for Greens functions is the solution of Poisson's equation in electrostatics. In this context the Greens function is, roughly speaking, the potential due to a point charge.

§3.2. Electrostatic Green's Function.

Suppose we want to find the electric potential due to a charge distribution. The mathematics of electromagnetism is based on the four coupled differential equations known as Maxwell's Equations. For a static electric field only one of those equations is needed, which is

$$\nabla \cdot \vec{E}(x, y, z) = -4\pi\rho(x, y, z) \tag{3.2.4}$$

The electric potential is a defined quantity rather than a measured quantity. Every point in space is considered to have a potential energy equivalent to the work performed if a point charge were placed in that position. You can visualize a point charge brought from infinity to the present position while work is performed against the electric field in its surroundings. Conversely, the "surrounding" electric field, $\vec{E}$, is define the gradient of the electric potential, and written.

$$\vec{E}(\vec{r}) = -\nabla U(\vec{r}) \tag{3.2.5}$$

When we substitute this definition of $\vec{E}$ in equation (3.2.4), the Maxwell's equation for the electric field, we obtain the definition of *Poisson's equation*

$$\nabla^2 U(x, y, z) = -4\pi\rho(x, y, z) \tag{3.2.6}$$

which is the basic equation for the electric potential. The solution of Poisson's equation is the well known inverse distance formula for the potential given by the following equation

$$U(\vec{r}) = \int_{V'} \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3\vec{r}' \tag{3.2.7}$$

The potential formula in equation (3.2.7) is, however, just an idealization for the case of "a charge in free space." The problem comes about when boundaries are introduced into a physical model. Boundaries change the potential that otherwise would have been given simply by (3.2.7). A boundary might become polarized by the surrounding electric field, and these induced dipole charges would then produce their own fields which interact with the ambient field. Boundaries comprised of conducting surfaces produce a similar effect due to the free charges induced, and these charges produce a secondary field that interacts with the ambient field. In other words, equation (3.2.7) is not enough for practical work.

If the polarization charge and the induced free charge distributions could be known in advance then expressions of the form of equation (3.2.7) could be used to compute the resulting, effective electric potential formula for a model. This is because the potential at any point is the sum of the potentials resulting from all free charges in the configuration. The dipoles and induced charges could each be interpreted as equivalent isolated charges and their potentials calculated using formula (3.2.7).

But you must already know the answer to the problem before you can obtain knowledge of the potentials induced by the boundary. This makes the solution of equation (3.2.4) a bigger problem than simply applying a superposition of equation (3.2.7) everywhere in the model. Problems with electrostatic fields are therefore solved by transforming the field problem into a potential problem. With equation (3.2.6) as an inhomogeneous, second-order differential equation, the field sources are the inhomogeneous term, and the solution constants depend on the potentials specified on the boundaries. This solution procedure is known in mathematical physics texts as solving *Laplace's equation*.

The approach most commonly taught to students of advanced physics is separation of variables on equation (3.2.6) in which known potentials are given on surfaces that lie in the direction of the (orthogonal) coordinate directions. But few practical problems can be solved with this method.

On the other hand, the finite element technique can be used to solve almost any geometric configuration, and the model construction requires very little understanding of the physics. The differential equation is transformed into a matrix equation by linearizing the continuum Laplacian operator into discrete, coupled Laplacian equations at each model node. But finite element models need uncommonly large matrices on a computer. There are several other practical difficulties with finite element modeling which offset the simplicity in theory. Models become unstable in regions of weak field coupling because the matrices become nearly singular there. Solution accuracy suffers when widely different geometric scales are involved, or widely different material coefficients.

The Green's function technique has been largely a curiosity in mathematical physics. It has not found practical use because you must first solve the boundary value problem before you can find the Green's function. Therefore, Green's functions have not received the attention in numerical mathematics that other methods have been given. But the Green's function technique deserves attention because a general formula exists which separately gives answers for each of the effects of source charges and boundary values.

To get an idea of why the Green's function technique is different from ordinary Laplace's equation solutions, consider the integral in equation (3.2.7) again. This time, look at the integrand as two factors: the source charge density, which is the function shown with the Greek letter ρ, and *another factor*.

$$\int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} d^3\vec{r}' = \int \rho(\vec{r}') \cdot \frac{1}{|\vec{r}-\vec{r}'|} d^3\vec{r}' = \int \rho(\vec{r}') G(\vec{r},\vec{r}') d^3\vec{r}' \tag{3.2.8}$$

If we look at the integral with the source charge density factored out the remaining factor contains only the inverse distance formula. This factor is considered to contain all the geometric information in the solution and is called the *propagator*. It is, in fact, the Green's function for a charge distribution with no boundaries present (boundaries located at infinity). That was easy, but the problem becomes difficult when boundaries are introduced. Then the inverse distance formula can no longer be used for the Green's function.

To demonstrate the effect of induced effects on the boundaries a similar factorization can, in principle, be written by merely inserting integrals over the hypothetical induced polarization charges. But to find the actual values of those induced charges on a dielectric we must solve a simultaneous problem–the work done to place free charges in the configuration and the work needed to polarize the surface, with a reference to a boundary value. A fixed potential on a boundary is equivalent to the presence of an equipotential surface with free charge. Again, a simultaneous problem must be solved with reference to the boundary value. Can we write an integral in which the boundary potential, not the boundary charge, is given?

§3.3. The Magic Rule.

A formula exists which gives the potential at any observation point due to any distribution of free charges and the specification of boundary potential values. Every student of mathematical physics is shown the Green's function magic rule:

$$U(\vec{r}) = \int \rho(\vec{r}') G(\vec{r},\vec{r}') dV' - \frac{1}{4\pi} \int U(\vec{r}') \frac{\partial G(\vec{r},\vec{r}')}{\partial n'} dS' \tag{3.3.9}$$

It depends on knowledge of a Green's function and the derivative of the Green's function on the boundary.

The magic rule is derived from Green's Theorem as a sequence of vector calculus operations. Given two functions $U(x,y,z)$ and $G(x,y,z)$, take the product U times the gradient of G and the product G times the gradient of U,

$$U(\vec{r})\nabla G(\vec{r})$$
$$G(\vec{r})\nabla U(\vec{r})$$

Then apply the divergence operation to both these expressions

$$\nabla \cdot (U(\vec{r})\nabla G(\vec{r})) = U\nabla^2 G + \nabla U \cdot \nabla G$$

and

$$\nabla \cdot (G(\vec{r})\nabla U(\vec{r})) = G\nabla^2 U + \nabla G \cdot \nabla U$$

and subtract the second from the first to obtain

$$\nabla \cdot (U\nabla G - G\nabla U) = U\nabla^2 G + G\nabla^2 U \tag{3.3.10}$$

The next step is to integrate both sides of this equation over the volume containing the sources and boundaries. The integration of the left-hand side is easily transformed by using the Divergence Theorem formula and results in an integration over the boundary surfaces. Then we obtain

$$\int U\nabla^2 G dV' - \int G\nabla^2 U dV' = \oint U\frac{\partial G}{\partial n'}dS' - \oint G\frac{\partial U}{\partial n'}dS' \tag{3.3.11}$$

Continuing to turn the crank of vector calculus we make substitutions for the volume integrals in order to obtain a formula for the potential. It can be seen that the integrand of the second integral on the left-hand side contains the divergence of the potential. We can, therefore, use equation (3.2.6)

$$\nabla^2 U(x, y, z) = -4\pi\rho(x, y, z)$$

as an identity and substitute the divergence of the potential with the source charge density function.

$$\int G\nabla^2 U dV' = -\int G(4\pi\rho)dV' \tag{3.3.12}$$

We can see that the right-hand side of this transformation will become the Green's function integral of equation (3.2.7). But how should we interpret the first integral on the left-hand side of (3.3.11)? In other words, what can we substitute for the divergence of G? The purpose of this substitution will be to replace the integral with the symbolic placeholder for the solution, $U(x, y, z)$.

The integral substitution is carried out by means of the definition equation (3.1.3) applied to Poisson's equation

$$\nabla^2 G(\vec{r}, \vec{r}') = -4\pi\delta(\vec{r} - \vec{r}') \tag{3.3.13}$$

We are saying that the divergence of this function G is equivalent to the Dirac delta function shown on the right-hand side. The defining property of the Dirac delta function is that it transforms, or simplifies, an integral into a function according to the rule

$$\int f(\vec{r}')\delta(\vec{r}-\vec{r}')dV' = f(\vec{r}) \tag{3.3.14}$$

With the Dirac delta function, and equation (3.3.12), the integral containing the divergence of G will be reduced to

$$\int U(\vec{r}')\nabla^2 G(\vec{r},\vec{r}')dV' = -4\pi U(\vec{r}) \tag{3.3.15}$$

Now using equation (3.3.12) and equation (3.3.15) substituted into equation (3.3.11) we have, after solving for the potential, U,

$$U(\vec{r}) = \int \rho(\vec{r}')G(\vec{r},\vec{r}')dV' +$$

$$\frac{1}{4\pi}\int G(\vec{r},\vec{r}')\frac{\partial U}{\partial n'}dS' - \frac{1}{4\pi}\int U(\vec{r}')\frac{\partial G(\vec{r},\vec{r}')}{\partial n'}dS' \tag{3.3.16}$$

This formula is called *the Green's function magic rule*, because it completely solves the electrostatics potential problem with free charge sources and potentials given on the boundaries. As mentioned before, the problem now is to find a function $G(\vec{r},\vec{r}')$ which satisfies equation (3.3.16). The two surface integrals are mutually exclusive (at least at the same location). The presence of both integrals (at the same location) is an over specification of the problem. From that condition follows a most important deduction: that when a potential value is given on a boundary we require that the function G goes to zero on the boundary so that the other surface integral will vanish.

Actually, the first surface integral in (3.3.16) is an irrelevant artifact of the derivation (3.3.10) through (3.3.16). The gradient of the potential on the boundary is just another representation of free charge. A particular model can be reconfigured such that all free charges are included within the volume integration. But the first surface integral is retained for discussions of Neumann boundary conditions.

Then the electric potential, for practical purposes, can be completely solved using the formula

$$U(\vec{r}) = \int \rho(\vec{r}')G(\vec{r},\vec{r}')dV' - \frac{1}{4\pi}\int U(\vec{r}')\frac{\partial G(\vec{r},\vec{r}')}{\partial n'}dS' \tag{3.3.17}$$

The function $U(\vec{r})$ is the electric potential observed at the point $\vec{r}$ due to the integration of infinitesimal charges (the volume integral) and boundary potentials (the surface integral) at the locations $\vec{r}'$. The integrations are carried out over regions that contain the sources. The Green's function is a "propagator," that is, a function which determines the change in potential over the region as a result of the geometry of the problem. The Green's function must be deduced for each new configuration of boundaries. When the boundary integral of (3.3.17) is absent the Green's function is the simple inverse distance function, but when the boundary integral is present it is not. The Green's function must take values of zero on the boundary, and this condition provides the beginning for a technique for calculating a Green's function definition.

The derivation of the magic rule as presented above provides no obvious hint as to how a Green's function can be found. Approaches to finding the Green's function focus on the identity (3.1.3),

$$\mathcal{L}\cdot G(x,x') = -\delta(x-x')$$

and the property that the Green's function is defined to be zero on the boundary. See the standard mathematical physics textbooks for a few techniques on actually finding a Green's function for a particular geometry. There are typically two solutions found: the one dimensional Poisson's equation and a free space version for the three-dimensional Poisson's equation.

Chapter 4

A Simple Explanation of Green's Functions

§4.1. A Three Point Diagram.

The current chapter is an attempt to examine the Green's function magic rule more closely than the standard textbooks have done. For one thing, the derivation of the magic rule from Green's theorem is typically performed as a purely algebraic operation without reference to diagrams. Then again, there has been very little attention to presenting a unified technique for constructing Green's functions applied to various geometries.

To begin, we will consider three points in one dimension: A, B and C.

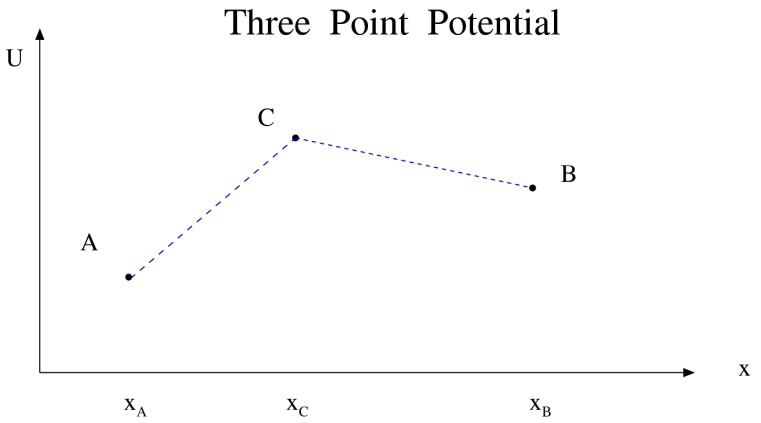

Figure 4.1.

Suppose we want to solve for the potential at point C due to potential values at points A and B. But, and we assert this *a priori*, this is equivalent to saying that the potential at point C is the solution to Laplace's equation (or Poisson's equation).

$$\frac{d^2U}{dx^2} = -f(x) \tag{4.1.1}$$

Using a finite difference approximation for the second derivative,

$$\frac{1}{\Delta x}\left(\left.\frac{dU}{dx}\right|_B - \left.\frac{dU}{dx}\right|_A\right) = -f_C \tag{4.1.2}$$

and further

$$\left.\frac{dU}{dx}\right|_B - \left.\frac{dU}{dx}\right|_A = -f_C \Delta x \tag{4.1.3}$$

Substituting a "charge" for the right-hand side,

$$f_C \Delta x = Q_C \tag{4.1.4}$$

and continue with the finite difference approximation by differencing the first-order derivatives.

$$\frac{U_B - U_C}{\Delta x} - \frac{U_C - U_B}{\Delta x} = -Q_C \tag{4.1.5}$$

With the objective of solving for U at C, multiply by the interval Δx

$$(U_B - U_C) - (U_C - U_B) = -Q_C \Delta x \tag{4.1.6}$$

and solving for the value of U at $x = c$, the answer is

$$U_C = \frac{U_A + U_B}{2} + \frac{\Delta x}{2} Q_C \tag{4.1.7}$$

Equation (4.1.7) is clearing suggestive of the Green's function magic rule, with U_C the calculated potential due to boundary values at A and B and a source at C. And looking again at equation (4.1.3), it resembles the Divergence theorem identity (or Gauss's Law). If equation (4.1.7) is any hint at the nature of the magic rule, then it says that we must integrate Poisson's equation twice and solve for U at the source location.

Suppose that the term in Q_C on the right-hand side of equation (4.1.7) were omitted. That would be equivalent to setting $f(x)$ to zero, that is, solving the homogeneous differential equation. For the case of the homogeneous differential equation the three-point potential diagram simplifies to a straight line potential between A and B

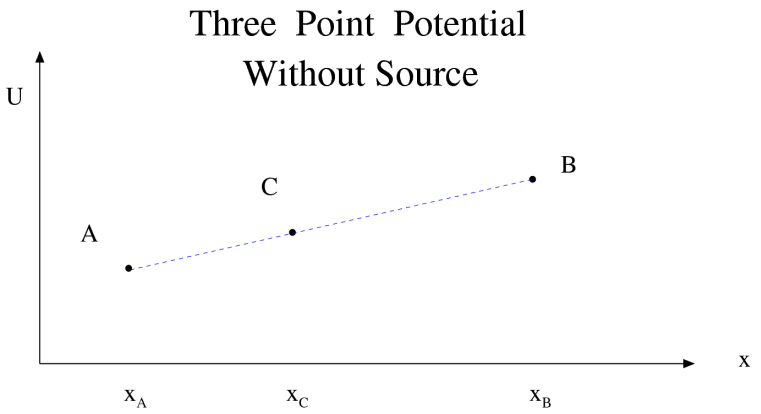

Figure 4.2.

Now this simple difference between figures Figure 4.1.1 and Figure 4.1.2 illustrates an important observation, that the charge is proportional to the difference in slope of the potential at point C. Or, from a physical

perspective it may be that the charge *causes* the difference in slope. This observation provides a hint to the relationship between the source term and the solution at a point. The second derivative in equation (4.1.1) is just the 1-dimensional form of the divergence (of the gradient). In two or more dimensions the effect of the source cannot be so simple reduced to a change in slope, but the 1-dimensional picture gives the fundamental idea:

REMARK 4.1.1.

The source is that which changes the slope of the function U, and to make the change of slope agree with the values at the boundary the value of U at the source location must be different from what it would have been without the source present.

Another simple, but important, observation is made by removing the source term from equation (4.1.7):

$$U_C = \frac{U_A + U_B}{2} \tag{4.1.8}$$

That is, for the homogeneous case the potential at the center is the average of the boundary values. But the most intriguing observation (guess why) is that *the solution, U, at the source location is finite.*

The solution in equation (4.1.7) was over simplified for the sake of starting the discussion. Important information was discarded or deliberately overlooked in the derivation which we must now restore. For one thing, the point separations need not be constant. If we restore the locations x_A, x_B, and x_C, then the solution becomes

$$U_C = \frac{x_C - x_A}{x_B - x_A} U_B + \frac{x_B - x_C}{x_B - x_A} U_A + \frac{(x_C - x_A)(x_B - x_C)}{x_B - x_A} Q_C \tag{4.1.9}$$

The expression in (4.1.9) now shows that the potential contribution from the boundary values depends on the point separation. Each term corresponding to a boundary value has a coordinate relation to the source term that resembles the differentiation of a function. A comparison of (4.1.9) with the magic rule suggests a correlation between the factor of Q_C in and a Green's function.

Our three-point potential picture in figure Figure 4.1.1 presents the observation point coinciding with the source point. We must now add more information to the picture so that source and observation point are different. If we select some arbitrary point, x, on the picture, where there is no source applied, then the potential U at x must lie along one of the straight lines in the figure. Suppose that we choose to observe the potential on the left side of the source point, as illustrated in figure.

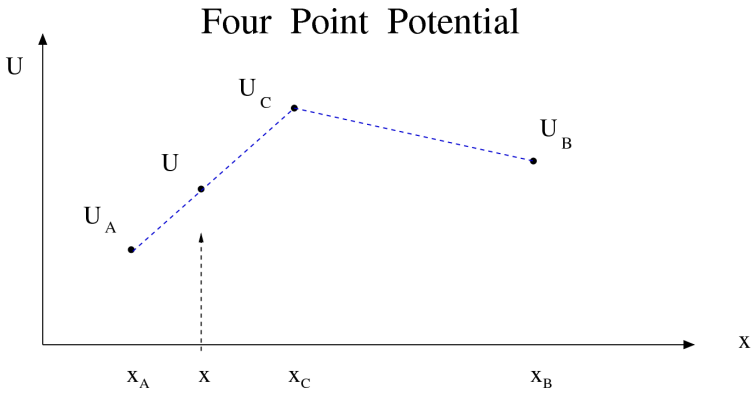

Figure 4.3.

We can simply interpolate the value of U at x from the geometry we have already constructed. A function $U(x)$ can be created as a straight-line expression between points A and C. What if we visualize figure Figure 4.1.3 as the superposition of two different three-point potential pictures: the first containing points A, $U(x)$, and C and the second containing points $U(x)$, C, and B. But now the right boundary of the first triad, point C, is not fixed. Similarly, the left boundary of the second triad, $U(x)$, is not fixed. So the values at $U(x)$ and C can be solved as a system of two equations in two unknowns.

$$\frac{U_B - U_C}{x_B - x_C} - \frac{U_C - U(x)}{x_C - x} = -Q_C$$

$$\frac{U_C - U(x)}{x_C - x} - \frac{U(x) - U_A}{x - x_A} = 0 \tag{4.1.10}$$

Now solving for U at the new location x the answer is

$$\frac{(x_B - x_C)(x - x_A)}{x_B - x_A} Q_C + \frac{U_B - U_A}{x_B - x_A} x + \frac{U_A x_B - U_B x_A}{x_B - x_A} \tag{4.1.11}$$

Notice that the two equations in (4.1.10) are interchangeable except that one point coincides with a charge, and the other does not. In other words, there is a reciprocity between the observation point x and the source point C. If we interchange the labels on points $U(x)$ and C this is equivalent to putting the observation point on the other side of the charge point.

$$\frac{U(x) - U_C}{x - x_C} - \frac{U_C - U_A}{x_C - x_A} = -Q_x$$

$$\frac{U_B - U(x)}{x_B - x} - \frac{U(x) - U_C}{x - x_C} = 0 \tag{4.1.12}$$

Again, solving for U at x we get

$$\frac{(x_B - x)(x_C - x_A)}{x_B - x_A} Q_x + \frac{U_B - U_A}{x_B - x_A} x + \frac{U_A x_B - U_B x_A}{x_B - x_A} \tag{4.1.13}$$

§4.2. The One-Dimensional Solution.

The formulas (4.1.11) and (4.1.13) give the value of U on either side of the source charge position. Notice that these two formulas look almost identical except that the labels on x and x_C are interchanged. Why must there be two formulas? We have just seen above that there are two formulas because we solved for the potential at two different locations. In other words, what was left unspoken in the above discussion is that there was an implied grid over which the points must lie. We started with a finite difference. Then, as the hand is quicker than the eye, we inserted a continuous-valued variable, x, into the picture. The transformation from discreteness to a continuum involves discarding information–in this case we are discarding grid labels.

Now convince yourself that with the above substitutions that expression (4.1.11) can be rewritten as

$$G_1(x, x') = \frac{L - x'}{L} x \tag{4.2.14}$$

Using the same substitutions in expression (4.1.13) we get

$$G_2(x, x') = \frac{x'}{L}(L - x) \tag{4.2.15}$$

Equation (4.2.14) and (4.2.15) can be recognized as the two parts of the one- dimensional Green's function shown in most mathematical physics textbooks. The functions $G_1(x, x')$ and $G_2(x, x')$ give the potential contributions for sources to the left side and right side of the observation point, respectively.

To jump from equations (4.1.11) and (4.1.13) to (4.2.14) and (4.2.15) seems as simple as a substitution of variables, but it better classified as a transformation from an infinitesimal domain to a global, continuous domain. That transformation has resulted in a function that represents the relative change in potential at a relative position on the x axis. It will be observed that any constant potential can be added to it, and its domain may be translated to any position. This circumstance makes clear a well-known property of Green's functions, namely that *The Green's function must be zero on the boundary because it is a relative change in potential.*

To demonstrate this, suppose we select a point charge at the "jth" location x_j, and an observation point to its left. This configuration corresponds to (4.2.14). But we know that at the location x any potential may by added, say U_0, that is a solution to Laplace's equation. But the potential U_0 may itself be decomposed into a sum of potentials, say, U_a and U_b.

$$U(x) = G_1(x, x_j)q_j dx + U_a + U_b$$

where q_j is the charge density in the region dx. Also, it may be that U_a is itself the potential resulting from a neighboring point charge at the $j-1$ location, x_{j-1}. Similarly, the additive potential U_b may be the result of another point charge on the right at the location $j+1$. Then the potential can be written

$$U(x) = G_2(x, x_{j-1})q_{j-1}dx + G_1(x, x_j)q_j dx + G_1(x, x_{j+1})q_{j+1}dx$$

In fact, for an arbitrary configuration of point charges–some that may be zero–the additive potential could be a summation of Green's function contributions on either side of the observation point,

$$U(x) = \sum_k G_2(x, x_k)q_k dx + \sum_j G_1(x, x_j)q_j dx \tag{4.2.16}$$

Note that equation (4.2.16) does not include the boundary conditions for the potential problem. Each of the functions $G(x, x')$ were defined to be zero at the boundary. In the limit as the number of point charges goes to infinity, and the intervals dx tend to zero the sum in (4.2.16) is replaced by an integration

$$U(x) = \int_{x_A}^{x} G_2(x, x')\lambda(x')dx' + \int_{x}^{x_B} G_1(x, x')\lambda(x')dx' \tag{4.2.17}$$

We have seen how a simple, three point potential diagram allowed us to capture the basic idea underlying Green's Theorem as well as providing a means for actually deriving a Green's Function. When we combined two three-point diagrams, one of them enclosing a source charge and the other enclosing an arbitrary point, we produced a propagation from point x' to x. Because of this we may choose to highlight the following observation, that

REMARK 4.2.2.

We solve for the potential at the source point, and we solve for the potential at the observation point. The two solutions are dependent on each other, being derived from overlapping three-point diagrams. Their simultaneous solution produces an expression for one in terms of the other.

As mentioned above, we skipped the details of "jumping" from the small, infinitesimal domain to the macroscopic world of continuous-valued functions. And so the defining property for Green's functions,

$$\mathcal{L} \cdot G(x, x') = -\delta(x - x') \tag{4.2.18}$$

did not appear in the above derivation. The Dirac delta function is the consequence that the Green's function satisfied Laplace's equation everywhere in the integration domain except the location of the source, where the divergence (of the gradient) is undefined. At the location of the source the gradient is discontinuous. In the derivation from a three-point potential we have simply ignored that fact by stating "there is a source at point $x = C$." A point is a mathematical object which has location but no spacial extent. But here is one of the underlying principles of the Green's function approach. The Green's function is the response due to a source with no spacial extent. From another perspective it appears that we have deliberately extracted information from a particular response function (information about extent) to be restored by the integration. This is the meaning of the discontinuity in the gradient which is encountered in the formal study of Green's functions.

For the sake of completeness, the textbook derivation of one-dimensional Green's functions is listed below. Although it appears somewhat truncated, it is consistent with the usual space allocated to Green's functions in the literature.

(1) Consider 2 regions on the one-dimension domain between the boundaries to be divided by the (infinitesimal) source.

(2) Use the homogeneous solution as a separate solution on either side of the source location.

(3) Set the solution in the left region to be zero at the left boundary, and set the right-hand side solution to be zero at the right boundary.

(4) Match the two solution functions at the source location, which enforces the continuity of the function.

(5) Match the derivatives of the two solution functions at the source location, given that the source location is at a discontinuity of the first derivative.

It should be observed that the above standard treatment incorporates the definition of equation (4.2.18), *and* that the Green's function is zero at the boundary. But the standard derivation leaves out the issue of the surface term in the Green's function magic rule. Of course, in one dimension the surface term is just the weighted sum of the boundary values. It is left for us to produce a clearer concept of the relationship between the boundary term and the volume term.

Consider again the three-point potential diagram and the associated finite difference solution equation (4.1.9). We have seen that the weight factor on $f(x)$ is not really equivalent to a Green's function, because the source location, x', and observation location, x, are coincident at $x = x_C$. But equation (4.1.9) does, however, incorporate the geometry of the configuration. And, since in one dimension the potential is finite at the source location, x', it will be clearer to use (4.1.9) and say that the $x = x'$, and

$$U' = \frac{x' - x_A}{x_B - x_A} U_B + \frac{x_B - x'}{x_B - x_A} U_A + \frac{(x' - x_A)(x_B - x')}{x_B - x_A} Q' \tag{4.2.19}$$

It can immediately be seen that the coefficients of the boundary terms constitute the weights of a weighted sum which sum to unity. The factor on the inhomogeneous term is comprised of the multiplication of both the numerators of the boundary factors. Recalling that an *nth* degree polynomial having real zeros can be written in point-zeros form as

$$P_n(x) = (x - x_1)(x - x_2)\dots(x - x_n)$$

it can be seen that the inhomogeneous factor is *a polynomial whose zeros are the boundary points.* As noted before, the weights on the boundary terms appear to be related to the inhomogeneous term as derivatives. In particular, the factor on U_B is equal to the derivative with respect to x_B of the weight on $f(x)$, and similarly for the weight on U_A.

§4.3. Visual Integration of the Boundary.

But instead of examining only the algebra, let's draw a picture of the relationship between the boundary weights and the pseudo Green's function, i.e., the inhomogeneous factor. The relationship we wish to illustrate is that the pseudo Green's function is a composition of the boundary weights. But rather than describe the boundary weights as derivatives of the inhomogeneous weight, let us attempt the inverse and say that the pseudo Green's function is obtained by separate integrations of the the boundary functions.

In other words, since the propagator in the surface integral of the magic rule is calculated as the derivative of the Greens function evaluated at the boundary point x_b,

$$\frac{\partial G(x, x')}{\partial x'}\Big|_{x' \to x_b}$$

The inverse would be to integrate the Green's function gradient from the boundary inward towards the source location,

$$\int_{x_b}^{x'} \frac{\partial G(x, \xi)}{\partial x} d\xi \to G(x, x')$$

Consider first the homogeneous equivalent of (4.2.19),

$$U' = \frac{x' - x_A}{x_B - x_A} U_B + \frac{x_B - x'}{x_B - x_A} U_A \tag{4.3.20}$$

Considered as a continuous-valued function of x', U' is a linear polynomial function of x',

$$U(x') = ax' + b \tag{4.3.21}$$

Then the relationship between equations (4.3.20) and (4.3.21) is that the linear polynomial in (4.3.21) is decomposed into *influence functions* in (4.3.20). This is illustrated with the following diagram, where $L_A(x')$ and $L_B(x')$ are influence functions.

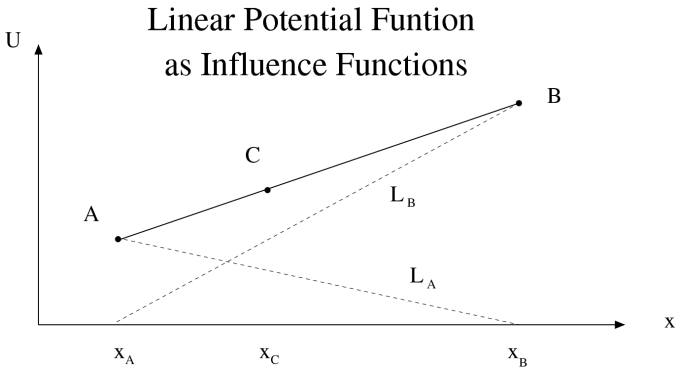

Figure 4.1.

Note that both $L_A(x')$ and $L_B(x')$ are rescaled to show their contributions to the solution. Each is defined to have values between zero and unity.

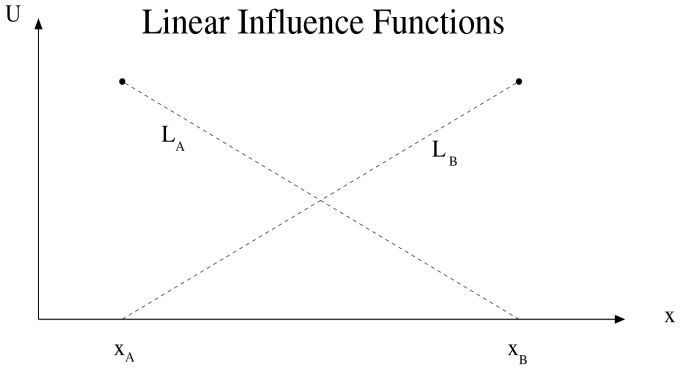

Figure 4.2.

One way to view the inhomogeneous, three-point potential is to present it as two homogeneous, two-point potentials back to back.

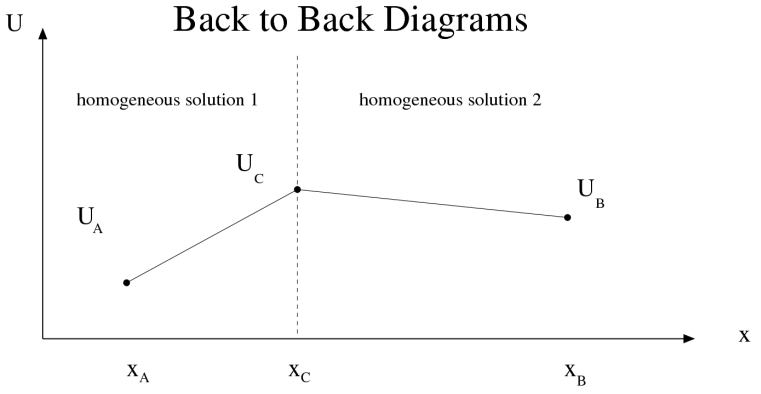

Figure 4.3.

From this point of view the source position is the right-hand boundary for the first problem and the left-hand boundary for the second problem. We have seen that the potential is finite at the source position, and so we can specify an unknown potential, U_C, at the source. And this is consistent with equation (4.2.19), which shows that the source term contributes a change in potential at x' that results in U'.

Now let's try to illustrate the inverse of the differentiation that produces $dG(x,x')/dx'$ from $G(x,x')$ by a "visual integration" of one of the boundary weights, say $L_A(x')$ from the boundary to an interior location. Imaging that we can grasp the boundary potential point at $x = x_A$ and drag it to the left to x'.

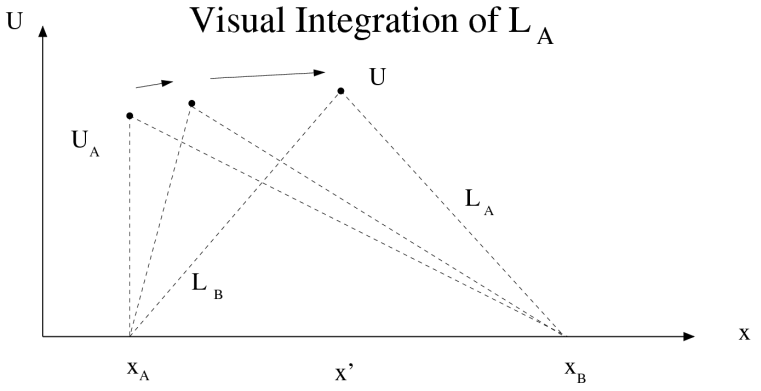

Figure 4.4.

The picture is supposed to illustrate the integration as the inverse of differentiation *that restores the factor* $(x - x_a)$.

$$\int_{x_A}^{x'} \frac{x_B - x'}{x_B - x_A} dx = \frac{(x_B - x')(x' - x_A)}{x_B - x_A} \tag{4.3.22}$$

Now convince yourself that the differentiation that produces $dG(x,x')/dx$ from $G(x,x')$ is the act of reversing the direction of the arrows in Figure 4.3.4. This is to suggest the following relationship between the pseudo Green's function and the boundary weights, that

REMARK 4.3.3.

The pseudo Green's function is the synthesis of all the boundary influence functions.

and

REMARK 4.3.4.

The differentiation of the pseudo Green's function at boundary location x_0 is equivalent to moving the source point, x', to x_0 and factoring out the dependence on x_0.

§4.4. A 2D Equivalent of Visual Integration.

The concept of visual integration of the boundary can be extended to a two-dimensional diagram, although it must be understood that it is only an approximation, valid only in the small, differential domain.

Suppose that the two-dimensional Poisson's equation

$$\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2} = -f(x,y) \tag{4.4.23}$$

is approximately integrated over the infinitesimal region of a triangle with the linear, homogeneous solution

$$U(x,y) = ax + by + c \tag{4.4.24}$$

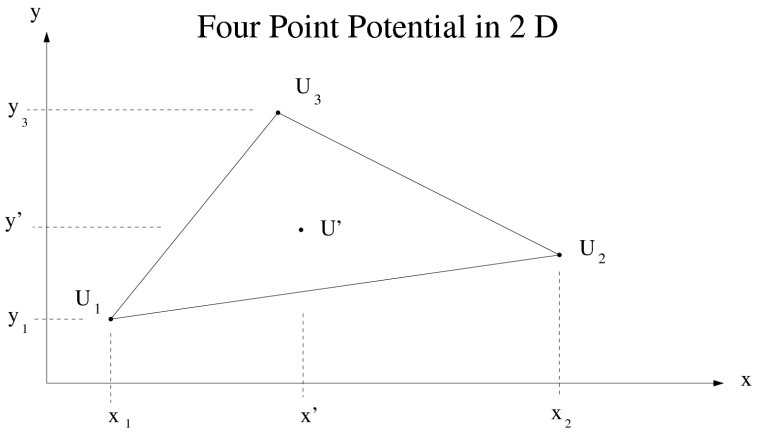

Figure 4.1.

The procedure for approximating the boundary value solution will be discussed in the later chapter on numerical Green's functions. The solution for a triangle will also be a weighted sum of the vertex points and their corresponding influence functions

$$U(x,y) = L_1(x,y)U_1 + L_2(x,y)U_2 + L_3(x,y)U_3 \tag{4.4.25}$$

In analogy with Figure 4.3.4 the visual integration of one of the boundary functions L_j should result in the pseudo Green's function as the equivalent inverse of dG/dx'. Again, imagine that we can grasp the boundary point at, say, point 1 and drag it into the triangle to point x', y'. The idea is illustrated in the following figure.

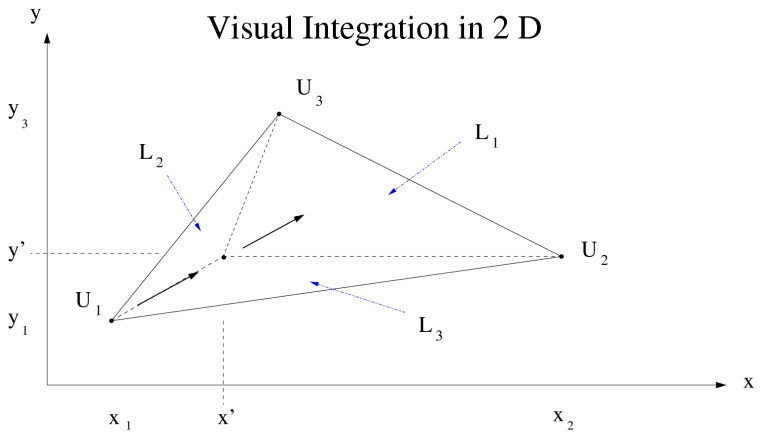

Figure 4.2.

Visually, it is seen that three different homogeneous (approximate) solutions are generated around *the inhomogeneous location* x', y' as it moves from the boundary into the "volume" of the infinitesimal element. Seen another way, there are now three homogeneous partitions of the element in which the source location, x', y', is the boundary point for each of the partitions.

The visual integration shown above is equivalent to a path integration that introduces a factor or $x' - x_0$, say, for an integration in x or $y' - y_0$ for an integration in y to the original influence function. The exact form of the factor produced by the path integration will be some function $A(x', x_0, y', y_0)$. Similarly, the integration from any boundary point, x_k, y_k to x', y' will produce a factor $A(x', x_k, y', y_k)$. The Green's function approximation, or at least the pseudo Green's function would then appear to be the product of all N factors that result from the N integrations, up to a multiplicative constant.

$$G(x', y') \rightarrow \mathcal{C} \prod_{k}^{N} A_k(x', x_k, y', y_k) \tag{4.4.26}$$

To reverse the procedure and obtain any one of the boundary influence functions the gradient of $G(x', y')$ must be taken and x', y' taken to the boundary point, x_0, y_0. It can be seen that the result of that gradient would be to divide out the factor $A(x', x_0, y', y_0)$ from $G(x', y')$. The resulting gradient then incorporates the necessary properties of an influence function in that it goes to zero at all the other boundary points. Since the influence functions from which it was integrated were polynomials, equation (4.4.26) is a polynomial whose zeros are *all* the boundary points.

§4.5. Inserting the Propagation Component.

But an Nth-degree polynomial we still not be a Green's function until we separate the observation point, x, from the source point, x'. Returning to the four-point potential in the triangle, the observation point is some location in one of the three partitions caused by the presence of the source at x', y'.

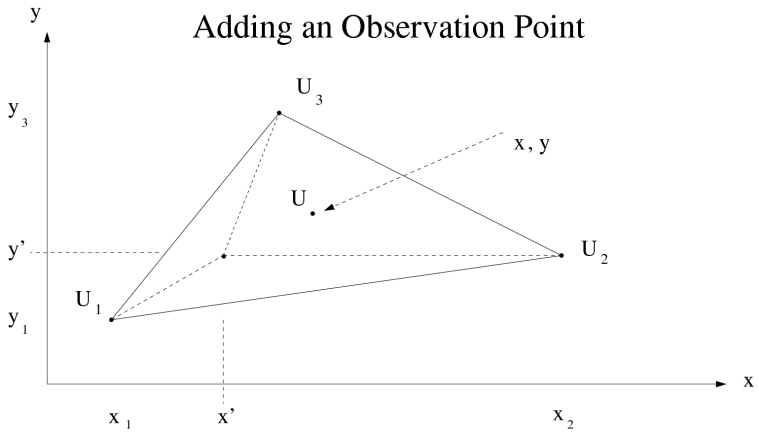

Figure 4.1.

It is clear from the diagram that the potential U at x, y is the solution of the homogeneous problem framed by one of the partitions, that is, whose solution is given by one of the influence functions *whose non-zero point is now anchored at the source location.* When we view the diagram as having been split into separate homogeneous partitions then we see that solving for the Green's function is another way of converting the source into the potential value at x', y' that matches the solutions in each of the homogeneous partitions.

However, solving for the potential at the observation x, y point within the selected partition involves knowing the potential at x', y'. If we return to the one-dimensional problem solved in the physics classroom we can see that this is accomplished by

(1) Matching the slope of the homogeneous solution at the source point x' (there difference equal to -1), and

(2) Matching the homogeneous solution at x'.

Note that the step in (1) above is equivalent to the Gauss Law integral in the small region around the source point. In other words, we impose the condition that the *curvature* (i.e., the divergence) of the potential in the neighborhood of the source charge, Q, is equal to $-Q$. But since we are factoring out Q (to be restored in the Green's function integration) we set the curvature to -1.

But to understand this matching condition it is instructive to return again to the three-point potential diagram. In the one-dimensional problem the potential at x' is that results from the change in slope and the boundary values. The matching condition in one dimension is equivalent to

$$\frac{\partial U}{\partial x}\Big|_{x'+} - \frac{\partial U}{\partial x}\Big|_{x'-} = -1 \tag{4.5.27}$$

which is the act of matching the gradient in the x direction between the two partitions in x. Then with N partitions (say, triangles) the N gradients should be matched at the source point, x', y'.

$$\sum_{k}^{N} \left(\frac{\partial U_k}{\partial x} + \frac{\partial U_k}{\partial y} \right) = -1 \tag{4.5.28}$$

The simultaneous solution of (4.5.28) and the continuity of $U(x', y')$ is used, as in the one-dimensional case, to set the coefficients in the functions $U_k(x, y)$.

But another approach may be used for inserting the observation point. If the pseudo Green's function can be obtained, then it may be used to get the value of the potential at the source point x', y'. In fact, it should be emphasized that

REMARK 4.5.5.

The pseudo Green's function, $G(x', y')$, *is* the value of the potential at x', y'.

Then, take a candidate homogeneous solution, $U_k(x, y)$, from one of the "partitions" of the figure which will contain arbitrary coefficients. The coefficients will be resolved by matching the solution to its value at x', y',

$$U_k(x', y') = G(x', y')$$

And solving for the unknown coefficients the answer, $G(x, x', y, y')$ is obtained.

§4.6. Relationship to General Green's Functions.

Green's functions for two and three dimensions are known for the special case of *free space* configurations. That means that the boundaries are assumed to be infinitely far away. The well-known two and three dimensional Green's functions can be found in Arfken, *Mathematical Methods for Physicists.*

The two-dimensional Green's function is

$$G(\vec{\rho}, \vec{\rho}') = -\frac{1}{2\pi} \ln |\vec{\rho} - \vec{\rho}'| \tag{4.6.29}$$

The three-dimensional Green's function is

$$G(\vec{r}, \vec{r}') = \frac{1}{4\pi} \frac{1}{|\vec{r} - \vec{r}'|} \tag{4.6.30}$$

But how can the simple picture of Green's functions that was illustrated with the three-point diagram be used to derive the above formulas? Consider the three-dimensional formula as an illustration that can be similarly applied to the two dimensional case. For simplicity, let the location of the source point, $\vec{r}'$, be at the origin so that $\vec{r}' = 0$. Also, the multiplier $1/2\pi$ is a dimensional factor that depends on the chosen units system, so it may be set to unity for simplicity. Then the Green's function is rewritten

$$G(r, 0) = \frac{1}{r} \tag{4.6.31}$$

There is no angular dependence since the Green's function is not a function of the spherical polar variables θ and ϕ. Now recall the one-dimensional, three-point potential diagram from Figure 4.1.1.

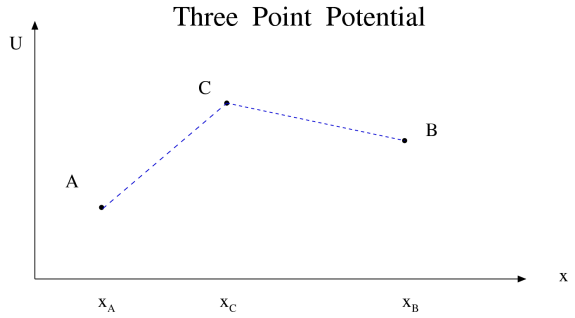

Figure 4.1.

But the major difference between the linear picture in the $x-y$ plane and the spacial configuration in r is that the curves between points A and C, and between points C and B, will not be straight lines. The derivation in one dimension, between equations (4.1.2) and (4.1.13), conceals an important property of the problem. The potential gradients between the points are simplified forms of *flux tubes*. It is not actually the gradient that vanished for the homogeneous equation but the gradient multiplied by the cross-sectional area of a virtual surface.

The cross-sectional area of the flux tube is readily seen when we perform the finite difference of the three-dimensional Poisson's equation in x, y, and z.

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = -f(x,y,z) \tag{4.6.32}$$

$$\frac{1}{\Delta x}\Big(\frac{\partial U}{\partial x}_B - \frac{\partial U}{\partial x}_A\Big) + \frac{1}{\Delta y}\Big(\frac{\partial U}{\partial y}_D - \frac{\partial U}{\partial y}_C\Big) + \frac{1}{\Delta z}\Big(\frac{\partial U}{\partial z}_F - \frac{\partial U}{\partial z}_E\Big) = -f'$$

Multiplying through by $\Delta x \Delta y \Delta z$ we get

$$\Big(\frac{\partial U}{\partial x}_B - \frac{\partial U}{\partial x}_A\Big)\Delta y \Delta z + \Big(\frac{\partial U}{\partial y}_D - \frac{\partial U}{\partial y}_C\Big)\Delta z \Delta x + \Big(\frac{\partial U}{\partial z}_F - \frac{\partial U}{\partial z}_E\Big)\Delta x \Delta y = -f' \Delta x \Delta y \Delta z \tag{4.6.33}$$

Here the finite difference replacements are too crude to incorporate all the essential information, because the cross sections $\Delta y \Delta z$ can vary along the x direction, and similarly for the other cross sections. In other words, each gradient term in (4.6.33) should have a unique cross section multiplier.

The consequence of having flux tubes between the points instead of mere gradients is that the varying of the cross sections forces the gradients to vary between the points (because the flux is conserved between the points). Consequently, the curves between the points are in general not linear. The definition of the three point potential must now be that the flux changes discontinuously at the source point. Consider the coordinate r as a series of concentric shells surrounding $r = 0$. Then the source point corresponds to a shell of "charge" at coordinate $r = r'$.

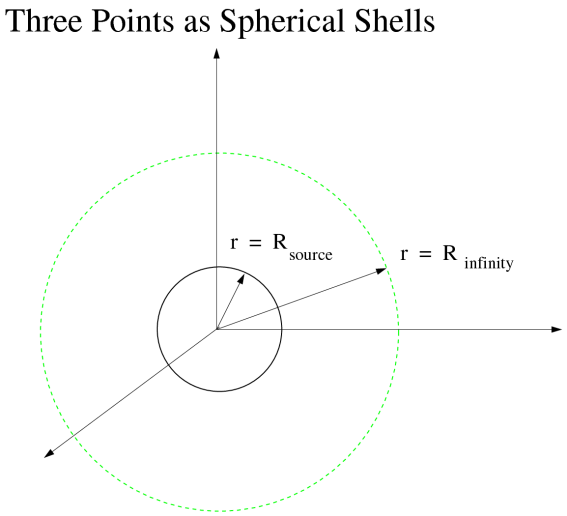

Figure 4.2.

The three-point potential in the radial coordinate r is shown below. The point r_A is at the origin, the point r_C is the source location $r = r'$, and the point r_B is the exterior boundary point, $r = R_\infty$.

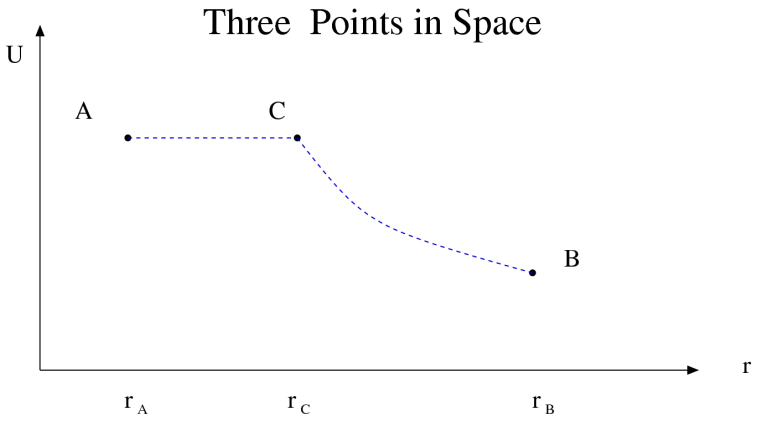

Figure 4.3.

The curve in the three-point potential between points C and B corresponds to a flux tube between the source shell and the infinity shell. The curve between points A and C is a flux tube within the source shell. But the region within a shell of constant potential is itself a region of constant potential, so the gradient is zero.

We have seen that the three-point potential is another way of describing the "pseudo Green's function" as shown previously. And we have seen that to transform the pseudo Green's function into a Green's function we must insert the propagation component, where the propagation is obtained merely by solving for the potential at some point within a partition of the pseudo Green's function (see Section 4.5). The appropriate partition for this form of the pseudo Green's function is between points r_C and r_B. The flux is conserved between A and B, so the following equation holds:

$$\mathcal{F} = \Delta A_C \frac{dU}{dr}\Big|_C = \Delta A_B \frac{dU}{dr}\Big|_B = const. \tag{4.6.34}$$

We saw how the solution of the simple three-point diagram in x resulted from two integrations, although we cheated in the integrations by merely substituting differences. The above expression in equation (4.6.34) is the result of the first integration, and it may be noted that it is equivalent to the Gauss Law integral. The second integration produces the potential, U, at the observation point r between points r_C and r_B. That is, we must calculate the work required to move a test charge from the shell at r_B inward, against the force of the electric field present, to r.

$$U(r) = -\int_{r_B}^{r} \vec{E} dr = -\int_{r_B}^{r} \frac{dU}{dr} dr \tag{4.6.35}$$

We already have an expression for dU/dr from expression (4.6.34). Because AdU/dr is constant along the flux tube, and A is the spherical polar differential element $r^2 d\Omega$,

$$\begin{aligned} r^2 \frac{dU}{dr} &= const \\ \frac{dU}{dr} &= \frac{const}{r^2} \end{aligned} \tag{4.6.36}$$

Then substituting (4.6.36) into (4.6.35) the answer is obtained,

$$U(r) = -\int_{r_B}^{r} \frac{const}{r'^2} dr' = \frac{const}{r}\Big|_{r_B}^{r} = \frac{1}{r} - \frac{1}{r_B} \tag{4.6.37}$$

The term $1/r_B$ is present because it represents the work performed to bring the test charge inward from infinity to the location r_B. The action that must be taken to make the solution equivalent to a *free space* Green's function is that the boundary at r_B is extended to "infinity." Obviously, in the limit as r_B becomes arbitrarily large,

$$\lim_{r_B \to \infty} \left(\frac{1}{r} - \frac{1}{r_B} \right) = \frac{1}{r} \tag{4.6.38}$$

so that the boundary at r_B can be ignored.

It can be seen that the solution above for $U(r)$ is still not the free space Green's function because the inner shell at r_C is still present. In other words, the solution $U(r)$ does not yet satisfy

$$\nabla^2 G(r, 0) = -\delta(r) \tag{4.6.39}$$

For that the inner shell which contains the source charge at r_C must shrink to (almost) zero. Recall that the flux is a constant. As we let the coordinate tend to zero, holding the flux constant, the charge density must become arbitrarily large near $r = 0$. Then we obtain the result that

$$G(r, 0) = \frac{1}{r} \tag{4.6.40}$$

satisfies the homogeneous Laplace's equation everywhere except at the location of the source at $r \to 0$ where it is undefined, and so this agrees with the defining statement (4.6.39).

Chapter 5

Numerical Green's Functions

§5.1. The Pseudo Green's Function Approximation.

As discussed back in Chapter 1, the archetypal boundary value problem is Laplace's equation in the homogeneous case,

$$\frac{\partial^2 U}{\partial x^2} = 0 \tag{5.1.1}$$

and in the inhomogeneous case, Poisson's equation,

$$\frac{\partial^2 U}{\partial x^2} = -f(x) \tag{5.1.2}$$

This is the simplest second-order equation because the differential equation has only the highest derivative on the left-hand side. As was presented in equation (1.2.5), the numerical approximation for Poisson's equation, derived from the linearized differential in a small region about the solution point, is essentially a weighted sum of the "boundary" points in the homogeneous case.

$$U(x') = \sum_{j=1}^{n} L_j(x) U_j \tag{5.1.3}$$

In the homogeneous case, a term is added which is the forcing function, $f(x)$, from the differential equation, evaluated at the solution point weighted by an approximation of the "pseudo Green's function," $P(x)$,

$$U(x') = \sum_{j=1}^{n} L_j(x) U_j + P(x') f(x') \tag{5.1.4}$$

This observation is simple given a finite difference approximation of Poisson's equation, and its solution. For example, the one-dimensional approximate solution of Poisson's equation in a small, differential region, is

$$U' = U_A \frac{(x_B - x')}{(x' - x_A) + (x_B - x')} + U_B \frac{(x' - x_A)}{(x' - x_A) + (x_B - x')} + f' \frac{(x_B - x')(x' - x_A)}{(x' - x_A) + (x_B - x')} (x_B - x_A) \tag{5.1.5}$$

Comparison of (5.1.4) and (5.1.5) shows that the latter provides an example of the weight functions. In this case it is obvious that the functions multiplying the boundary point, U_A and U_B, are the two-point, Lagrange interpolation functions.

But the inhomogeneous term must still be interpreted. By comparing (5.1.4) and (5.1.5) the correspondence is made that the $P(x)$ is related to the pseudo Green's function, $G(x)$.

$$P(x') = \frac{(x_B - x')(x' - x_A)}{(x' - x_A) + (x_B - x')}(x_B - x_A) = G_s(x')(x_B - x_A) \tag{5.1.6}$$

The factor $x_B - x_A$ can be seen as the corresponding, numerical, volume integration that appears in the Green's function magic rule. The factor $G_s(x')$ in (5.1.6) is the normalized product of the unnormalized factors appearing in each of the boundary weights. The normalization factor is the sum of the (unnormalized) boundary weights. If the unnormalized boundary weights are denoted by the symbols w_j, and the factors contained in the boundary weights are denoted by the symbol A_k, then the expression for each of the boundary weights must be

$$\begin{aligned} w_j &= \prod_{k \neq j} A_k \\ W_j &= \frac{w_j}{\sum_j w_j} = \frac{\prod_{k \neq j} A_k}{\sum_j w_j} \end{aligned} \tag{5.1.7}$$

The pseudo Green' function is then defined by expression that is a synthesis of the boundary weights,

$$G_s = \frac{\prod_k A_k}{\sum_j W_j} \tag{5.1.8}$$

It can be seen that the pseudo Green's functions is the product of *all* the factors in the set which comprise the unnormalized boundary weights.

In the two-dimensional equivalent problem the Poisson's equation

$$\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2} = -f(x, y) \tag{5.1.9}$$

the finite difference integration of the two second-order partial derivatives produces interpolation functions that align with the Cartesian coordinate axis

$$\begin{aligned} U' = U_A \Delta y \frac{(x_B - x')(y_D - y')(y' - y_C)}{\mathcal{N}} + U_B \Delta y \frac{(x' - x_A)(y_D - y')(y' - y_C)}{\mathcal{N}} + \\ U_C \Delta x \frac{(y_D - y')(x_B - x')(x' - x_A)}{\mathcal{N}} + U_D \Delta x \frac{(y' - y_C)(x_B - x')(x' - x_A)}{\mathcal{N}} + \\ f' \Delta x \Delta y \frac{(x_B - x')(x' - x_A)(y_D - y')(y' - y_C)}{\mathcal{N}} \end{aligned} \tag{5.1.10}$$

where the normalization factor $\mathcal{N}$ is the sum of the unnormalized weight factors on the boundaries,

$$\begin{aligned} \mathcal{N} = \Delta y(x_B - x')(y_D - y')(y' - y_C) + \Delta y(x' - x_A)(y_D - y')(y' - y_C) + \\ \Delta x(y_D - y')(x_B - x')(x' - x_A) + \Delta x(y' - y_C)(x_B - x')(x' - x_A) \end{aligned}$$

An examination of the factors appearing in (5.1.10) shows that the interpolation depends on the dimensions of opposing areas. Here the boundary points x_A, x_B, y_C, and y_D are at the midpoints of the figure, not at the vertices.

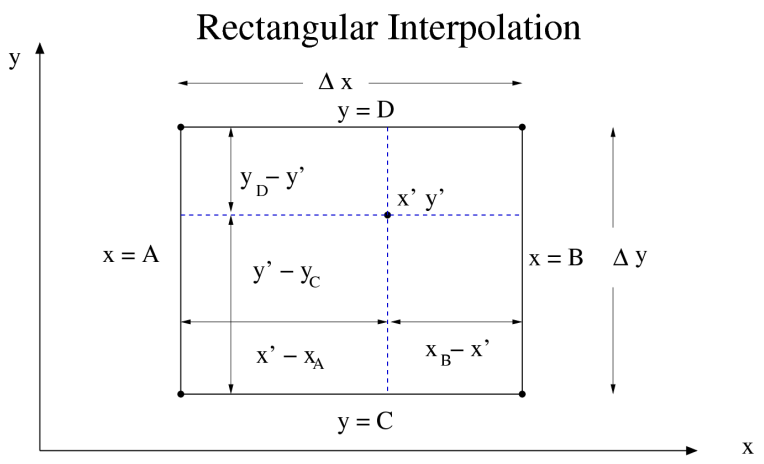

Figure 5.1.

From equation (5.1.10) it is seen that the factoring of the influence functions that weight the boundary points, and the factors in the pseudo Green's function, follows the algorithm shown in (5.1.7) and (5.1.8). The pseudo Green's function is a polynomial whose zeros are on the boundary points.

Comparison of the last term in (5.1.10),

$$f'\Delta x\Delta y\frac{(x_B-x')(x'-x_A)(y_D-y')(y'-y_C)}{\mathcal{N}} \tag{5.1.11}$$

with the pseudo Green's function,

$$\frac{(x_B-x')(x'-x_A)(y_D-y')(y'-y_C)}{\mathcal{N}} \tag{5.1.12}$$

shows the extra factors Δx and Δy, which are not present in the general polygon interpolations derived from the Vandermonde matrix or the opposing areas method (see Section 2.4, *2D Interpolation by Opposing Areas*). The question naturally arises how to interpret Δx and Δy with regard to the numerical weighted sum approach to integrating the differential equation.

The factor $\Delta x\Delta y$ appears to be equivalent to the volume integration in the Green's function magic rule. Likewise, the factors Δx and Δy on the boundary terms appear consistent with the surface integration in the magic rule. Another question arises as to the role of the magic rule integrations with respect to the numerical integration being performed here by the finite differencing. A hint that will help resolve this issue appears conspicuously in the normalization factor $\mathcal{N}$. Since Δx and Δy appear as factors in the sum of terms in $\mathcal{N}$ it follows that they must be incorporated into the weight factors, because the normalization is the sum of the weights.

Since the pseudo Green's function has been identified as the product of all the factors, one each, that appear in the weight functions it would be permissible to include $\Delta x\Delta y$ in the definition of the pseudo Green's function. For numerical purposes, the decision will ultimately depend on whether the model parameters are set to insert the density $f'(x',y')$ as the source samples or the finite quantity $Q' = f'\Delta x\Delta y$ as the source. However, if the weighted sum interpolation is generalized to non-orthogonal shaped elements, such

as Delauney triangles, and the method of opposing areas is used, the pseudo Green's function obtained will be the natural factor of f', not Q'.

Henceforth in this discussion it will be assumed that the general approach of interpolation by opposing areas will be used. Then the numerical solution of Poisson's equation will be taken from the definition shown in equation (1.2.5),

$$U(x', y') = \sum_{j=1}^{n} W_j(x', y')U_j + P(x', y')f(x') \tag{5.1.13}$$

The weight functions will be defined according to expression (5.1.7),

$$w_j = \prod_{k \neq j} A_k$$

$$W_j = \frac{w_j}{\sum_j w_j} = \frac{\prod_{k \neq j} A_k}{\sum_j w_j} \tag{5.1.14}$$

with the "power factor," $P(x', y')$ defined in accordance with expression (5.1.8),

$$P(x', y') = \frac{\prod_k A_k}{\sum_j W_j} \tag{5.1.15}$$

The formulas listed above represent the numerical integration of the simplest of all boundary-value differential equations. The work needed for generalizing the weighted sum approach for particular differential equations containing additional derivative terms will require modification of the weight factors and, consequently, of the power factor.

In summary, the Green's function magic rule was taken to represent the form of the numerical solution around any particular node in the finite difference model. It is seen that in numerical integration interpolation possesses similar properties to what propagation does in the global domain. The interpolations are constructed from finite differencing the derivatives, and it will be seen that with more derivatives added to the differential equation the interpolations will be modified thereby. The pseudo Green's function concept affords a method for incorporating the effect of the inhomogeneous term in the differential equation as a polynomial whose zeros are the boundary points of the *element* at a particular *node* of the finite difference model.

§5.2. The Induced Boundary Charge Method.

The most fundamental approach to finding a Green's function in electrostatic field theory is the Method of Images. With the Method of Images the boundary conditions are simulated by the placement of point charges at symmetric locations. The presence of a point charge in the vicinity of a conducting boundary causes equal and opposite charges to be induced on the surface of the boundary. The field lines on the conductor are perpendicular to the surface. The potential on the (grounded) surface is held at a fixed potential, which can be assigned a value of zero. On the other hand, it can be noticed that the mathematical configuration of two point charges results in a symmetry line between them where the potential and field lines have the same values that would occur on a conducting surface. The Method of Images is a technique for producing the same boundary conditions that occur on a conducting surface, that is, zero potential and perpendicular field lines, with an equivalent configuration of "image" charges.

The method of virtual charges on the boundary is NOT the familiar method of images. Virtual charges is a more brute force method than the method of images. Virtual charges are the actual charges that would be induced on the surface, not the imposition of a virtual image charge. The question is posed: what distribution of boundary charges can balance the effect of a point source charge and make the potential at the boundary zero? For a point charge at the source point, P_s, and a point on the boundary, P_b, what distribution of charges along the boundary makes the potential at P_b equal to zero? The answer is found by calculating the sum of the potentials at P_b due to the (unknown) charges on the boundary and the source charge. The sum is repeated at every point, P_b, along the boundary to obtain a set of N simultaneous equations in N unknowns: the N charges on the boundary. The values of the charges on the boundary is the solution which allows us to create a Green's function.

The Green's function is the potential sum, using the free-space potential formula for every source charge–the boundary charges and the "source" charge. The Green's function, however, is valid only for the fixed location of the point source charge that was used to find the boundary charges. The solution must then be repeated for each location of a source charge in the source distribution.

To be quite clear, this approach is none other than the elementary physics lessen whereupon a potential due to a distribution of point charges is calculated with a sum of free-space Green's functions. One of those point charges will be the "source" charge, and all the others are the induced surface charges. This is perhaps an iterative approach to finding the Green's function in the interior of a closed region: first treat all the charges as if they were in free space, then use the result to form a (single) Green's function that incorporates the boundary. It may be observed as well that this approach is related to what is called the boundary element method.

The above ideas are illustrated with the following mathematical expressions. The static electric potential at point $\vec{r} = x, y, z$ due to a charge distribution at $\vec{r}' = x', y', z'$ is typically written using the Green's function "magic rule"

$$\Phi(\vec{r}) = \int_{v'} G(\vec{r}; \vec{r}')\rho(\vec{r}')dv' \tag{5.2.16}$$

where $G(\vec{r}; \vec{r}')$ is the Green's function. In free-space the Green's function has the form, in 3 dimensions, of of

$$G(\vec{r}, \vec{r}') = \frac{1}{4\pi|\vec{r} - \vec{r}'|} \tag{5.2.17}$$

or in 2 dimensions

$$G(\vec{r}, \vec{r}') = \frac{1}{4\pi} \ln|\vec{r} - \vec{r}'| \tag{5.2.18}$$

To evaluate the integral (5.2.16) over the boundary region numerically we divide the integration region into

a grid, and (5.2.16) becomes a sum of integrations over each grid element. The potential at point $\vec{r}$ due to all the charges at points $\vec{r}_j'$, can be written

$$\Phi(\vec{r}) = \sum_j \bar{\Phi}_{ij} \tag{5.2.19}$$

where the $\bar{\Phi}_{ij}$ is the potential at i due to a charge distribution at j. Each term in the sum of (5.2.19) is an integration over a grid element. Let the charge distribution be constant over the (small) region of the grid element. Then the potential at location i due to a charge at location j is

$$\bar{\Phi}_{ij}(\vec{r}) = \int_{\Delta v} G_{ij}(\vec{r};\vec{r}')dv' \tag{5.2.20}$$

where v is a volume element approximating a surface element by making the dimension, δ, perpendicular to the surface very thin. Remember, we are not solving with the standard Green's function magical rule, so don't expect the familiar surface integral here. On a two-dimensional problem, say a rectangle, v will be an element around the point x,y on the boundary where the potential is observed, and v' will be at the point x', y' of the source point charge on the boundary. The elements will be taken along the line of the boundary, perhaps $\Delta v = \Delta x \cdot \delta$ or $\Delta v = \Delta y \cdot \delta$.

Now write the potential observed at the same location, v, due to a the "source" charge in the interior of the figure. This is the source charge in the usual sense of the word: it is the charge that usually appears under the volume integral term of the Green's function magic rule.

$$\bar{\Psi}_i(\vec{r}) = \int_{\Delta v} G_i(\vec{r};\vec{R})dV' \tag{5.2.21}$$

where the integral is taken over a small volume "locus", dV', around the unit source charge of constant density at $\vec{R}'$.

A system of N equations in N unknowns will be constructed. Each equation represents the sum of potentials at a particular point on the boundary. In order to achieve a potential of zero on the boundary each of these sums must be equal to zero. The system of equations is equivalent to a matrix equation in which the matrix contains the potentials calculated using the boundary charges as sources, and the source vector contains the potentials from the source charge. The solution vector will be the distribution of charges along the boundary that makes the potential there zero:

$$\begin{pmatrix} \bar{\Phi}_{11} & \bar{\Phi}_{12} & \cdots & \bar{\Phi}_{1N} \\ \bar{\Phi}_{21} & \bar{\Phi}_{22} & \cdots & \bar{\Phi}_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\Phi}_{N1} & \bar{\Phi}_{N2} & \cdots & \bar{\Phi}_{NN} \end{pmatrix} \begin{pmatrix} \bar{q}_1 \\ \bar{q}_2 \\ \vdots \\ \bar{q}_N \end{pmatrix} = \begin{pmatrix} -\bar{\Psi}_1 \\ -\bar{\Psi}_2 \\ \vdots \\ -\bar{\Psi}_N \end{pmatrix} \tag{5.2.22}$$

An example of the induced boundary charge method is shown in the figure below. The potential is held at zero at the boundary, and a "single" charge is introduced to the center of the square.

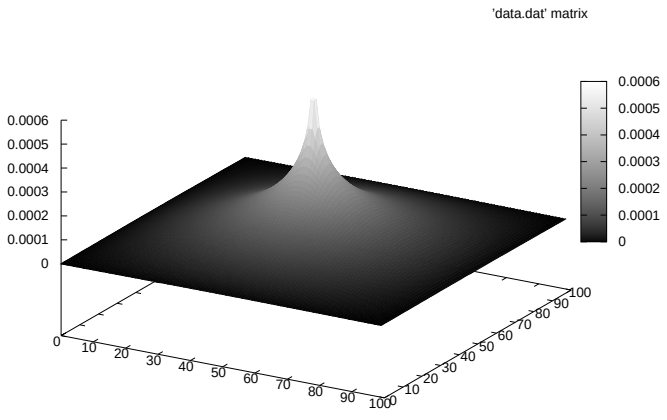

Figure 5.1.

The figure shown was obviously produced for a source charge with fixed location at the center. For point charges at other locations the matrix must be solved again and again. Also, it should be noted that the matrix becomes singular in the limit as the number of boundary charge samples becomes large. It is a simple observation that the singularity comes from the similarity between equations for adjacent charge locations. As the number of locations increases, the distance between them decreases, and one equation in the system of equations becomes nearly equal to the adjacent location's equation.